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Extraction of Vibration Characteristics of Cracked Structures Using Peridynamic and Bézier Numerical Modeling

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Abstract

Modeling the initiation and propagation of cracks in dynamic structures remains a persistent challenge within the Classical Continuum Mechanics (CCM) framework, primarily due to stress singularities at crack tips and the breakdown of spatial derivatives in the presence of displacement discontinuities. These difficulties are compounded by the strongly nonlinear nature of solid mechanics, which leads to stiff systems of timedomain ordinary differential equations and places severe restrictions on numerical stability and efficiency. Peridynamic (PD) theory, formulated as a nonlocal continuum framework, provides a natural remedy to these limitations by replacing local differential operators with integral expressions, thereby eliminating the requirement for spatial differentiability and enabling a unified description of deformation, damage initiation, and crack growth. Despite these advantages, the practical application of PD to dynamic fracture problems is still constrained by the choice of timeintegration scheme. Conventional explicit direct-integration methods are straightforward to implement but typically demand prohibitively small time steps to satisfy stability criteria, resulting in high computational cost. On the other hand, Adaptive Dynamic Relaxation (ADR) techniques are effective for accelerating convergence in quasi-static analyses but rely on artificial damping, which compromises their accuracy and physical fidelity in genuine dynamic simulations. In this work, a novel explicit time-integration strategy based on the Bézier numerical method is introduced for solving the peridynamic governing equations, referred to as the PD-Bézier approach. The method explicit the smooth approximation properties of Bézier polynomials to construct a stable and accurate explicit time update, enabling larger admissible time steps without sacrificing solution quality. The performance of the proposed PD-Bézier scheme is systematically assessed through comparative studies against conventional direct-integration and ADR methods. Numerical results demonstrate that the PD-Bézier formulation accurately captures the dynamic response of cracked structures, including natural frequencies and transient displacement fields, while maintaining computational efficiency and avoiding artificial damping effects. These findings indicate that the proposed approach offers a robust and effective alternative for dynamic peridynamic simulations and provides a reliable computational tool for the identification, analysis, and health monitoring of damaged mechanical and structural systems.

Keywords: Peridynamics, Computational Mechanics, Plane Vibration, Bézier Numerical Method

Introduction and Literature Review

Structural failure in components and plates, which are extensively used in the aerospace, automotive, and civil engineering industries, is frequently accompanied by vibrational phenomena. Excessive vibration can lead to rapid component wear, fatigue, connection loosening, and ultimately, sudden structural failure.Cracked structures exhibit distinctly different dynamic behavior compared to healthy structures. The presence of a crack causes a localized reduction in stiffness and an overall decrease in natural frequency values, alongside changes in the vibrational mode shapes. Investigating this altered vibrational behaviour is a cost-effective and rapid method for structural Damage Diagnosis and fault detection. Despite extensive research on the transverse (flexural) vibrations of plates, relatively little attention has been given to the longitudinal (in-plane) vibrations of rectangular plates containing cracks. This research specifically focuses on in-plane vibrations [1].

Fracture modeling within Classical Continuum Mechanics (CCM) is founded upon local relationships and differential equations. These methods face serious challenges when material discontinuities, such as cracks, are introduced. At the crack tip, stress fields become indeterminate due to the requirement for spatial derivatives, leading to the phenomenon of stress singularity. This necessitates the use of complex, external criteria to predict the initiation and path of crack growth.

Peridynamic (PD) theory, introduced by Silling in 2000 [2], is a non-local theory of solid mechanics developed to address the limitations of CCM in modelling discontinuities. By redefining differential spatial relationships into an integral form, PD eliminates spatial derivatives from the equation of motion. This formulation provides the capability to analyse damage in any discontinuity (crack or void) without requiring remeshing techniques.

Solving the PD equations, particularly in dynamic problems that require tracking the evolution of the response over real time (such as vibrations), necessitates a stable and accurate solution for a system of Ordinary Differential Equations (ODEs).

Conventional time-stepping methods, including explicit Direct Integration and Adaptive Dynamic Relaxation , possess limitations [3]. To overcome these constraints, the Bézier numerical method is proposed as an explicit and stable alternative. The Bézier method is a multi-step technique that offers higher stability and accuracy over larger time steps compared to traditional single-step methods. The synergy between the spatial modelling power of PD for cracks and the temporal efficiency of the Bézier method creates a complete and efficient computational solution for analyzing the dynamics of damaged structures. This paper focuses on the innovation that, to date, a complete simulation of free vibration in a cracked plate using a Bézier-based PD formulation has not been conducted[4] .

Theoretical Foundations of Peridynamic Modeling

The Bond-Based Peridynamics (BB-PD) formulation is the simplest and earliest version of PD theory. In BB-PD, the force between any two material points is exerted only along the direction of the bond vector [5]. At any instant, each point in the material specifies the location of a material particle, and an infinite number of material points (particles) constitute the continuous medium.

In the deformed configuration as shown in Figure 1, the mateial points $x_{(k)}$ and $x_{(j)}$ experience displacements $u_{(k)}$ and $u_{(j)}$, respectively. Their initial relative position vector $x_j - x_k$ is transformed into the post-deformation position vector $y_{(j)} - y_{(k)}$. The stretch between material points $x_{(k)}$ and $x_{(j)}$ is defined as:

$$s_{(k)(j)} = \frac{\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right|}{\left|x_{(j)} - x_{(k)}\right|} \quad 1$$

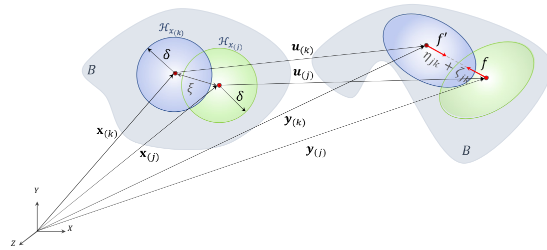

Figure 1 Interaction of material points in reference and deformed coordinates

In the reference configuration, every material point x interacts with other points x' within a finite region of radius δ(horizon). This region is known as the family (H_x). The relative position vector in the reference configuration is defined as $\xi = x' - x$, and the relative displacement between two points is $\eta = u(x',t) - u(x,t)$

The governing equation of motion in PD at time t for a material point x is expressed in integral form [6]:

$$\rho \ddot{u}(x,t) = \int_{\mathcal{H}_x} f(x,x',u,u',t)dv_{x'} + b(x,t) \qquad 2$$

In equation 2 , ρ is the mass density of the material, $\ddot{u}(x,t)$ is the acceleration, $b(x,t)$ is the external force density, and f is the internal traction force density per unit area of volume, applied from x' to x . The traction force f is a function of the relative displacement η and the reference relative position vector ξ . This force is defined through the micro-modulus constant c and the stretch s. The stretch s is defined as the relative change in bond length [7].
Peridynamic equations do not possess generalized analytical solutions. Consequently, numerical approaches are employed to compute the spatial and temporal integrals of the PD equations. To this end, equation 1 is rewritten in a discrete form:

$$\rho_k \ddot{u}_k = \sum_{j=1}^{NF_{(k)}} f_{kj}(\eta_{kj}, \xi_{kj}, t)V_{(j)} + b_{(j)} \qquad 3$$

In equation 3, ξ_{jk} is the initial bond length calculated from the relation $\xi_{jk} = \left|x_j - x_k\right|$, and η_{jk} is the relative displacement of the points, calculated $\eta_{jk} = \left|u_j - u_k\right|$. $NF_{(k)}$ is the number of points within the family of point k. The internal traction density between two points in the Bond-Based method can be obtained from the following relation:

$$f_{kj} = cs_{kj}\overrightarrow{e_{kj}} \qquad 4$$

In this relation, c is the Bond-Based peridynamics micro-modulus constant. $\vec{e}_{kj}$ is the unit vector in the direction of the line connecting the two material points [8]. To determine the material constant c in Bond-Based Peridynamics in terms of engineering material constants, the strain energy density of the material points is equated within the PD and CCM frameworks. This process is performed for two simple loading conditions, namely isotropic expansion and pure shear, which results in the following definitions:

$$c = \frac{2E}{A\delta^2} \quad 1D \quad , \quad c = \frac{12k}{\pi h \delta^3} \quad 2D \quad , \quad c = \frac{18k}{\pi \delta^4} \quad 3D \qquad 5$$

E and k are Young's modulus and the bulk modulus, respectively. Additionally, A and h are the element area and element thickness, respectively.
In PD, damage and fracture are modeled through bond breakage. When the stretch (s) of a bond exceeds its tolerance limit, the critical stretch (s_c), that bond irreversibly breaks and its traction force drops to zero. The value of s_c is calculated based on equating the PD strain energy with the critical strain energy release rate (G_c) in classical fracture mechanics [9].

Time Integration Algorithm and the Bézier Numerical Method

The Bézier method is a multi-step numerical technique for solving Ordinary Differential Equations (ODEs) in the time domain. It employs Bernstein polynomial parametric curves. These curves are defined using "control points" that influence the curve's path but do not necessarily lie on the curve itself [10]. In this approach, the temporal integral of the PD equation of motion is approximated at each step using control points. The main advantage of Bézier over single-step methods (such as Euler) is that by utilizing information from previous points $\left(y_n, y_{n-1}, \ldots\right)$ and not just the endpoints of the interval, it offers higher accuracy (lower truncation error) while simultaneously providing superior numerical stability. Multi-step Bézier methods, particularly the 2-step and 3-step formulations, are capable of maintaining numerical stability over larger time steps compared to explicit Direct Integration methods. This characteristic directly reduces the computational cost in long-term PD dynamic simulations [11].It has been shown that the 2-step and 3-step Bézier methods strike an optimal balance between the required accuracy and computational expense. Increasing the order of the Bézier method (e.g., to 4-step or higher) only marginally improves accuracy but drastically increases computational costs (solution time). Since the PD model is inherently time-consuming, selecting an optimized 3-step method preserves the main advantage of Bézier: the reduction of the overall dynamic solution time [10]. The formulas for the Bézier method perform time integration over a time step h based on the function values at previous time stages. For example, the three-stage (3-step) Bézier formula, which provides the necessary stability and accuracy for solving PD dynamic equations, is expressed as follows:

$$y_{n+1} = y_n + \frac{h}{12}\left(19 f_n - 8 f_{n-1} + f_{n-2}\right) \qquad 6$$

In this equation, $yn+1$ is the calculated displacement at the next step, y_n is the current displacement, and f_n is the function value at consecutive time steps. By using a second-degree polynomial function to approximate the temporal integral, this method is capable of tracking the system's dynamic evolution with higher stability and lower error compared to conventional methods.

Results and Validation

1. Modeling of a Rectangular Plate Under Free Vibration

Prior to modeling the cracked structure, the efficacy of the PD-Bézier model was validated for the elastic analysis of a healthy plate. A rectangular steel plate ($0.5\text{m}\times0.5\text{m}$)) with the following mechanical properties was modeled:

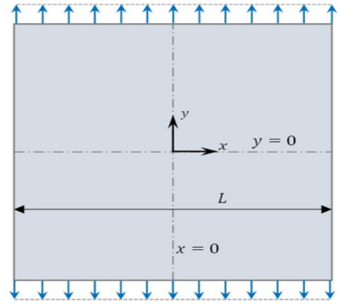

Figure 2 plane geometry under vibration

Table 1 Geometric and mechanical characteristics of rectangular plane

Parameter	Value	Unit
Young's Modulus(E)	192	G Pa
Poisson's Ratio (v)	0.33	-
Mass Density (ρ)	8000	kg/m^3
Length and Width (L,W)	0.5	m

We gave the sheet an initial displacement at the initial time and left it to vibrate freely, then recorded the displacement-time graph and then extracted the dominant amplitude frequency using the Fast Fourier Transform.

Initial Conditions:

$$u_y\left(y=\frac{-\ell}{2}, t=0\right)=-0.01mm \quad \& \quad u_y\left(y=\frac{\ell}{2}, t=0\right)=0.01mm \qquad 7$$

$$\dot{u}_x(t=0,x,y)=0 \;\; \& \;\; \dot{u}_y(t=0,x,y)=0$$

The displacement profile of a specific point on the plate over time showed a damped sinusoidal response, which confirms the physical nature of free vibration.

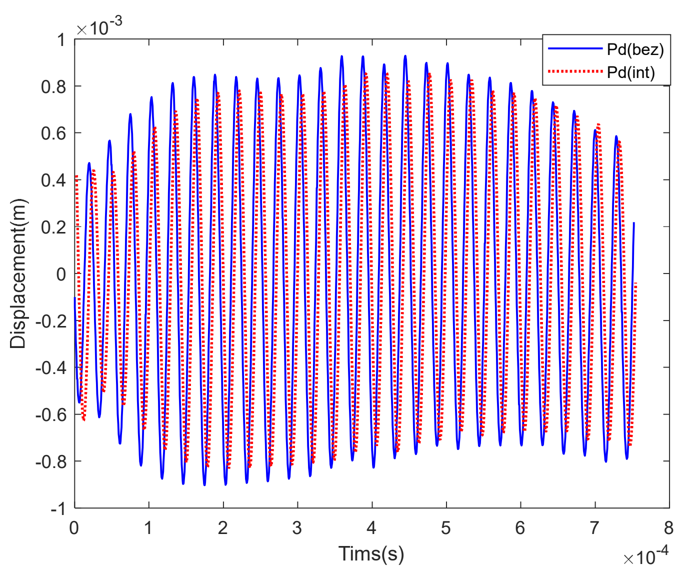

Figure 3 Comparison diagram of the displacement of a point on a rectangular plane under free vibration with respect to time using the Bézier and direct integral methods.

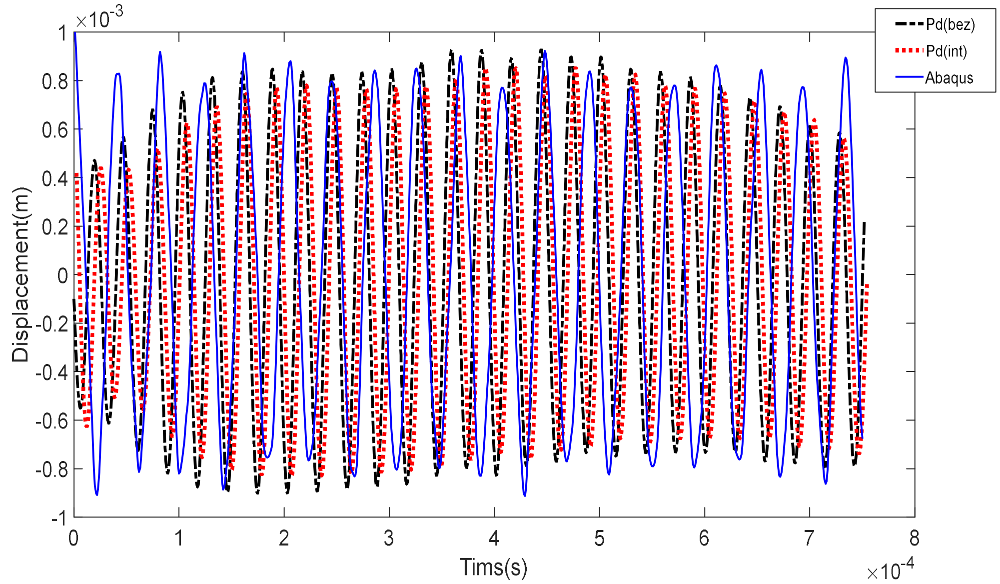

Figure 4 comparing the displacement diagram of a point on a rectangular plate under free vibration with respect to time using PD and FEM methods

The displacement-time vibration behaviour for this problem is plotted in the Figure 3 , Figure 4. By performing a Fourier Transform, its frequencies were extracted, and the results of the dynamic analysis are compared with FEM in the table.

Table 2 Comparison of First Mode Natural Frequencies of the Healthy Rectangular Plane

Abaqus (FEM)	PD (Bézier)	PD (Direct Integration)
35.17 MHz	33.25 MHz	33.9 MHz

The comparison results showed that the displacement response of the Bézier method closely aligns with the results obtained from the PD Direct Integration method. Furthermore, when compared against the Abaqus software (FEM), the PD-Bézier response demonstrates acceptable conformity in terms of oscillatory trend and damping.

2. Modeling of a Cracked Rectangular Plate Under Free Vibration

In this problem, while maintaining the geometry, boundary conditions, and mechanical properties, a 10 cm horizontal crack was introduced at the center of the plane. This crack is shown in the Figure 5.

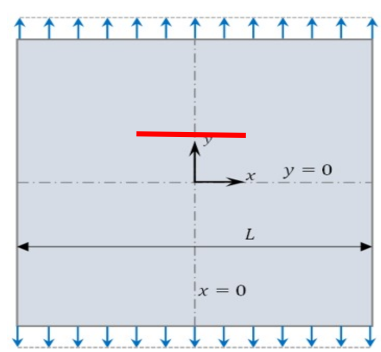

Figure 5 plane geometry under vibration with crack

Initial Conditions:

$$u_y\left(y=\frac{-\ell}{2},t=0\right)=-0.01mm \quad \& \quad u_y\left(y=\frac{\ell}{2},t=0\right)=0.01mm \tag{8}$$
$$\dot{u}_x(t=0,x,y)=0 \;\; \& \;\; \dot{u}_y(t=0,x,y)=0$$

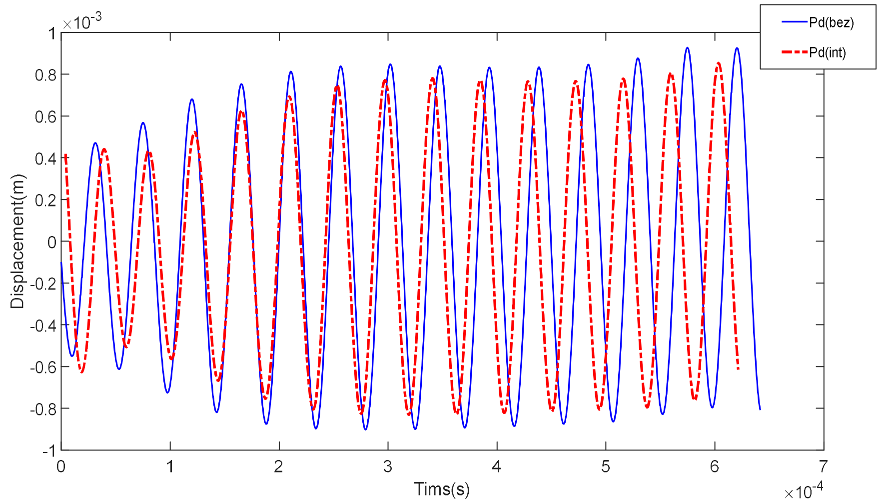

Figure 6 Comparison diagram of the displacement of a point on a rectangular plane under free vibration with a crack versus time using the Bézier and direct integral methods.

Table 3 Comparison of First Mode Natural Frequencies of the Healthy Rectangular Plane

Abaqus (FEM)	PD (Bézier)	PD (Direct Integration)
18.2 MHz	16.36 MHz	16.78 MHz

The damage behaviour of a plate containing a central horizontal crack was investigated using the Bézier-based and Direct Integration methods. The results, as shown in Figure 6, demonstrate the capability of both methods to model the displacement of material points in the plate due to free vibration. The difference between the results obtained with the FEM and PD methods is clearly visible in the graphs. This difference arises because the FEM represents damage by reducing the structure's global stiffness, whereas the peridynamics approach models damage as the local failure of individual bonds at the site of damage initiation, yielding a localized reduction in stiffness.

Conclusion

This study successfully developed the Peridynamic-Bézier (PD-Bézier) scheme, a novel and computationally efficient method for simulating the dynamic response of cracked structures. The core innovation is using the stable, explicit Bézier numerical method to integrate the equations of motion in Peridynamics, overcoming the limitations of conventional integrators.The method was validated against established techniques, accurately modelling both healthy structures and the physical effects of damage. The PD-Bézier scheme is presented as a valuable diagnostic tool for structural health monitoring.

Nomenclature

A	The cross-sectional area of the structure	$u_{\alpha(j)}$	α component of the displacement vector of the material point j

B	Continuum body	V_k	Volume of material point k
b_j	External force density vector	$x_{(j)}$	The initial position vector of a material point
f_{jk}	Internal force density vector	$y_{(j)}$	Deformed position vector of a material point
h	The thickness of the structur	$\Delta u_{\alpha jk}$	Relative displacement components of two material points j and k in the α direction
K	Micro-modulus matrix	$\Delta x, \Delta y$	Space between two adjacent material points in x and y directions
$N^F_{(j)}, N^M_{(j)}$	Number of bonds in fiber zone and matrix zone of point j	δ	Horizon size
t	Time	θ_{jk}	The angle of the shear bond
$u_{(j)}, \ddot{u}_{(j)}$	Displacement and acceleration vectors	μ_{jk}	Function to represent the state of interaction
ξ_0	The initial bond length of the smallest shear bond in the set of shear bonds with identical angles	ρ	Mass density
ξ_{jk}	The initial relative position vector between two points of j and k	σ	Stress matrix

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