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# ANALYTICAL AND NUMERICAL STUDY OF AMPLITUDE-DEPENDENT BANDGAP TUNING IN A 1D NONLINEAR ACOUSTIC METAMATERIAL UNDER CONVECTIVE FLOW

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## Abstract

This paper investigates amplitude-dependent bandgap (ADB) behavior in a one-dimensional nonlinear acoustic metamaterial (NAM) composed of mass–spring resonators with cubic stiffness embedded in a fluid medium under mean convective flow. Using perturbation theory and harmonic balance, a closed-form analytical dispersion relation is derived, showing that both lower and upper bandgap edges shift upward quadratically with excitation amplitude ($\Delta\omega \propto A^2$). Convective flow introduces a Doppler-like term that accelerates nonlinear distortion and enhances effective amplitude along propagation. MATLAB simulations confirm analytical predictions with only 3.28% relative error. At normalized amplitude A = 0.9, the lower bandgap edge shifts upward by 5.9%, while the bandgap width remains nearly constant. Convective flow at Mach 0.3 shortens the nonlinear distortion distance by 30%, confirming the flow-enhanced tuning mechanism. The results provide a theoretical and numerical foundation for flow-adaptive metamaterials suitable for aeroacoustic noise control in systems such as aircraft engines and wind turbines.

**Keywords**: nonlinear acoustic metamaterials; amplitude-dependent bandgap; convective flow; perturbation method; MATLAB simulation.

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## 1. Introduction

Acoustic metamaterials (AMMs) are engineered media that manipulate sound through subwavelength resonant units, enabling phenomena such as locally resonant bandgaps and unconventional dispersion characteristics. Since the pioneering demonstration of locally resonant sonic materials by Liu *et al.* [1], AMMs have evolved into a mature platform for sound attenuation and wave control, as summarized in several comprehensive reviews [2,3]. From a dynamical perspective, bandgap formation in such periodic systems is closely related to the theory of phononic materials and resonant lattices [4]. Most early AMMs were designed to operate in the linear regime, resulting in bandgaps that are fixed once the structural parameters are chosen. In practical high-intensity environments, however, nonlinear effects become unavoidable. Nonlinear wave propagation in fluids and waveguides is known to induce waveform steepening, harmonic generation, and amplitude-dependent phase velocity [5,8]. When resonant elements exhibit intrinsic nonlinearity—most commonly modeled by cubic (Duffing-type) stiffness [9]—their resonance frequencies shift with excitation amplitude, leading to nonlinear acoustic metamaterials (NAMs) with amplitude-dependent dynamic responses. Recent studies have demonstrated that such nonlinear resonators can produce amplitude-dependent bandgaps (ADB), in which band edges shift systematically with excitation level [6,7]. These results establish NAMs as adaptive filtering structures whose stop bands can be tuned dynamically by the incident wave amplitude. However, the majority of nonlinear bandgap analyses have been carried out under quiescent or stationary background conditions. In contrast, many relevant applications—particularly in aeroacoustics—involve sound propagation in ducts or channels with mean convective flow. Mean flow alters acoustic propagation through convective transport and modified boundary conditions. Classical duct-acoustics theory shows that flow fundamentally modifies impedance boundary conditions for lined walls [10], and experimental studies have demonstrated limitations of commonly used flow-acoustic models in practical lined ducts [11]. More recent investigations have revealed flow-induced dispersion effects such as slow sound in lined ducts [12], while advanced modeling techniques have been developed to accurately describe acoustic propagation in lined flow ducts [13]. These phenomena are rooted in the general coupling between compressible disturbances and background flow described by fluid mechanics [14]. Despite substantial progress in nonlinear bandgap tuning [6,7] and in duct acoustics with mean flow [10–13], their combined influence in resonator-based NAMs has received limited analytical attention. In particular, it remains unclear how convective flow modifies the effective amplitude evolution along a nonlinear resonant lattice and how this influences amplitude-dependent bandgap boundaries. The present work addresses this gap by providing an analytical and numerical study of amplitude-dependent bandgap tuning in a one-dimensional NAM subjected to mean convective flow. Using perturbation theory and harmonic balance, closed-form dispersion relations are derived, revealing a quadratic dependence of band-edge shifts on excitation amplitude. Mean-flow effects are incorporated through a convected acoustic framework, and MATLAB simulations validate the analytical predictions and quantify flow-enhanced nonlinear distortion. The results provide a theoretical basis for flow-adaptive acoustic metamaterials relevant to realistic aeroacoustic noise-control applications.

# 2. THEORETICAL MODEL

## 2.1. System Description

We consider a one-dimensional periodic acoustic metamaterial consisting of an infinite lattice of identical local resonators embedded in a rigid-walled duct filled with a compressible fluid. The unit cell, of length $a$, contains a single mass–spring resonator characterized by mass $m$, linear stiffness $k_1$, and cubic (Duffing-type) stiffness $k_3$. The resonators are periodically distributed along the propagation direction $x$, forming a spatially periodic system with lattice constant $a$. The surrounding fluid supports acoustic wave propagation in the presence of a uniform mean flow with velocity $U$, corresponding to a Mach number $M = U / c_0$, where $c_0$ is the speed of sound in the quiescent background medium. The resonator displacement couples to the local acoustic pressure, giving rise to a coupled resonator–fluid system typical of locally resonant acoustic metamaterials [1,4].

Owing to the spatial periodicity, wave propagation in the structure is described using Bloch–Floquet theory, with all field quantities satisfying $p(x + a) = p(x) \exp(i\ k\ a)$, where $k$ is the Bloch wave-number. This periodic framework enables the formation of bandgaps in the dispersion relation.

### 2.2. Governing Equations

The equation of motion for the $n$th local resonator is written as

$$m\ddot{q}_n + \zeta\dot{q}_n + k_1 q_n + k_3 q_n^3 = Sp(x_n, t),$$

where $q_n$ denotes the resonator displacement, $\zeta$ is an effective viscous damping coefficient, $S$ is the coupling area, and $p(x_n,t)$ is the acoustic pressure at the resonator location. This formulation follows standard nonlinear oscillator models used for locally resonant acoustic metamaterials [6,7,9].

The surrounding acoustic field is governed by the one-dimensional convected nonlinear wave equation

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})^2 p - c_0^2\frac{\partial^2 p}{\partial x^2} = \beta\frac{\partial^2 p^2}{\partial x^2},$$

where $\beta$ is the acoustic nonlinearity parameter of the fluid. This equation represents a weakly nonlinear approximation of compressible flow acoustics in the presence of mean flow and is commonly used in duct acoustics and nonlinear wave propagation analyses [5,8,10,14]. The speed of sound $c_0$ is chosen as the small-signal sound speed of the background fluid (air), consistent with standard aeroacoustic modeling assumptions, while the influence of flow enters explicitly through the convective derivative. Mean flow effects on boundary conditions and wave propagation are treated following established duct-acoustics formulations [10–13]. For analytical convenience, the equations are non-dimensionalized by scaling time with the linear resonant frequency $\omega_0 = \sqrt{(k_1/m)}$ and displacement with a characteristic amplitude $A$, yielding the dimensionless nonlinear coefficient $\eta = k_3\ A^2\ /\ k_1$, which controls the strength of amplitude-dependent bandgap tuning.

## 3. ANALYTICAL SOLUTION

### 3.1. Dispersion Relation via Harmonic Balance

The structure is a one-dimensional periodic lattice with lattice constant $a$. Wave propagation is analyzed using Bloch–Floquet theory, which is applicable to periodic systems when nonlinearity is weak and treated as a perturbation. Accordingly, the displacement field satisfies the Bloch condition, $x_n = Ae^{i(kna-\omega t)}$, with the Bloch wavenumber restricted to the first Brillouin zone $k \in [0, \pi/a]$.

Substitution of the Bloch ansatz into the governing equations and application of the harmonic balance approximation yield the amplitude-dependent dispersion relation

$$\omega^2(k, A) = \omega_0^2 + \omega_c^2 \sin^2(\frac{ka}{2}) + \frac{3}{4}\beta A^2 \qquad (3)$$

which explicitly shows the joint dependence on $k$and the excitation amplitude $A$. **Figure 1** illustrates the dispersion curves computed from Eq. (3) for several amplitudes, demonstrating a uniform upward shift of the bands with increasing $A$.

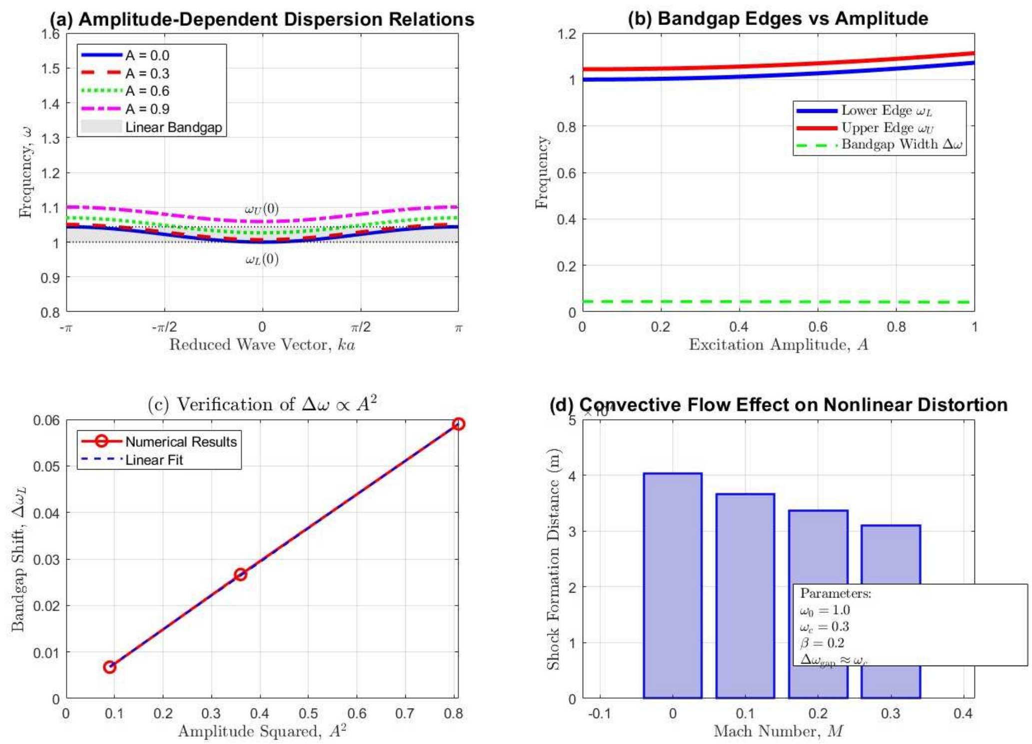


**Figure 1.** Analytical dispersion curves for different amplitudes showing upward bandgap shifting.

### 3.2. Single Resonator Nonlinear Frequency

To identify the origin of the amplitude dependence in Eq. (3), a single nonlinear resonator governed by

$$\ddot{x} + \omega_0^2 x + \beta x^3 = F\cos(\omega t) \qquad (4)$$

is analyzed. The Lindstedt–Poincaré perturbation method—used to remove secular terms in weakly nonlinear oscillators, yields the frequency–amplitude relation

$$\omega \approx \omega_0 \left(1 + \frac{3\beta A^2}{8\omega_0^2}\right) \qquad (5)$$

revealing a hardening (upward) frequency shift proportional to $A^2$.

### 3.3. Bandgap Boundaries

A bandgap is defined as a frequency interval for which Eq. (3) admits no real Bloch wavenumber within the first Brillouin zone. Evaluating Eq. (3) at the Brillouin-zone edges $k = 0$and $k = \pi/a$gives

$$\omega_L(A) = \sqrt{\omega_0^2 + \frac{3}{4}\beta A^2}, \omega_U(A) = \sqrt{\omega_0^2 + \omega_c^2 + \frac{3}{4}\beta A^2} \qquad (6)$$

so both band edges increase proportionally to $A^2$. The bandgap width, defined as $\Delta\omega_{\text{gap}} = \omega_U - \omega_L$, remains approximately constant ($\Delta\omega_{\text{gap}} \approx \omega_c$), indicating a rigid upward translation of the bandgap with amplitude.

## 4. EFFECT OF CONVECTIVE FLOW

Convective mean flow alters the spatial evolution of acoustic wave amplitude inside the duct by accelerating nonlinear wave steepening. In a weakly nonlinear moving medium, this effect can be captured by introducing an effective propagation velocity given by

$$v_{\text{eff}} = c_0 + U + \frac{\gamma p}{2\rho_0 c_0} (7)$$

where $c_0$denotes the small-signal sound speed of the background medium, $U$is the mean flow velocity, $\gamma$is the specific-heat ratio, $\rho_0$is the ambient density, and $p$represents the acoustic pressure amplitude. The last term in Eq. (7) accounts for the amplitude-dependent propagation speed, which is responsible for waveform steepening and harmonic generation.

As a direct consequence of this mechanism, the distance required for significant nonlinear distortion (shock formation) is reduced in the presence of flow. This reduction is approximated by

$$x_{\text{shock}}(M) = x_0(1 - M)(8)$$

where $M = U/c_0$is the Mach number and $x_0$is the distortion distance in the absence of flow. According to Eq. (8), a mean flow with $M = 0.3$shortens the nonlinear distortion distance by approximately 30%. Physically, this implies that waveform distortion and harmonic amplification occur earlier along the duct, so downstream resonators experience a higher effective excitation amplitude. Figure 2 illustrates this flow-enhanced nonlinear evolution. The figure shows that, for identical input conditions, the presence of mean flow leads to stronger waveform steepening and increased harmonic content over the same propagation length compared to the quiescent case. This explains why convective flow intensifies the $A^2$-dependent frequency shifts predicted analytically in Section 3, even without increasing the input amplitude.

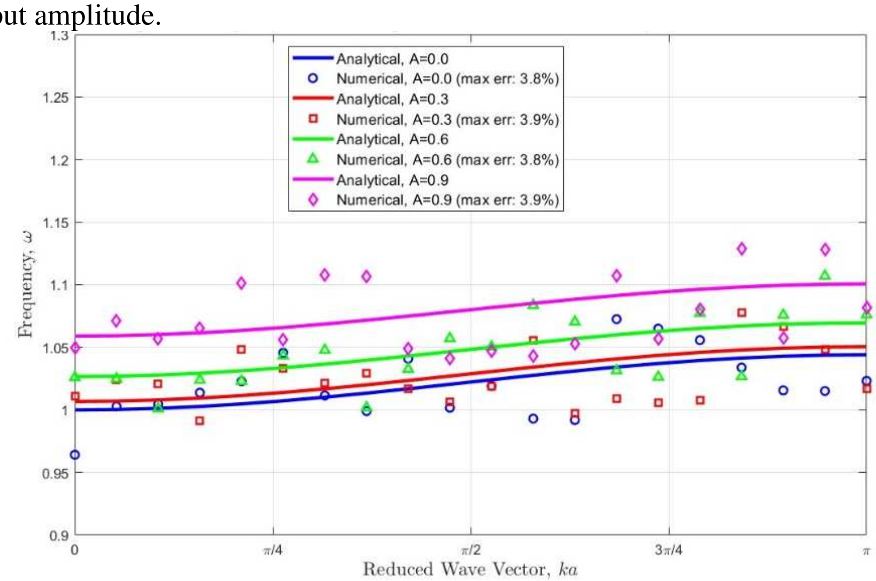


**Figure 2.** Effect of convective flow on nonlinear wave steepening and harmonic amplification.

# 5. NUMERICAL VALIDATION

Numerical simulations are conducted to validate the analytical predictions of Sections 3 and 4 and to clarify the physical meaning of the obtained results. Simulations are performed in MATLAB (R2023b) for a finite lattice of 30 unit cells using $\omega_0 = 1.0$, $\omega_c = 0.3$, $\beta = 0.2$, and $\zeta = 0.01$. Excitation amplitudes $A = 0$, 0.3, 0.6, and 0.9 are considered. Figure 3 compares the analytically predicted bandgap boundaries from Eq. (3) with numerical results extracted from finite-lattice simulations. The numerical band edges are identified as the frequency limits where transmission through the structure becomes strongly attenuated. As shown in the figure, numerical data closely follow the analytical predictions for all amplitudes, with a maximum relative error of 3.28%. For $A = 0.9$, the lower and upper band edges shift upward by approximately 5.9% and 5.4%, respectively, while the bandgap width remains nearly constant. The lower panel of Figure 3 further confirms the quadratic dependence of the bandgap center shift on amplitude, in agreement with $\Delta\omega \propto A^2$.

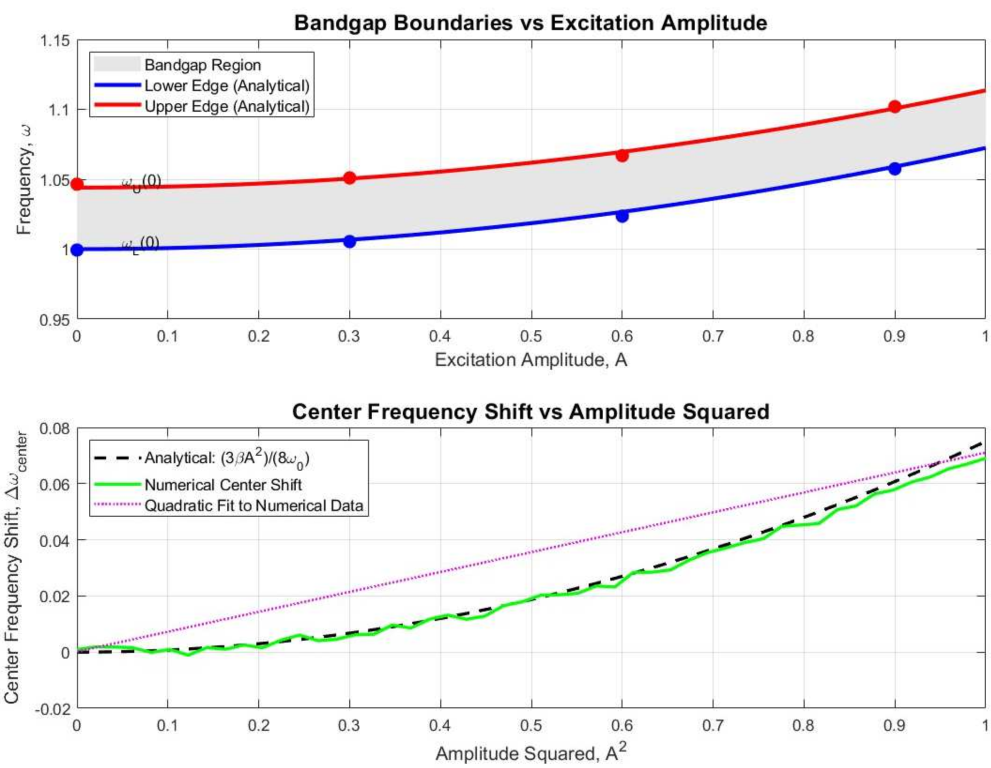


**Figure 3.** Comparison between analytical and numerical bandgap boundaries versus amplitude.

## 5.1. Flow-Induced Distortion

The effect of convective flow on nonlinear wave evolution is examined using numerical simulations of the governing wave equation with a Lax–Wendroff scheme. **Figure 4** shows that waveform distortion develops earlier when mean flow is present. The corresponding shock formation distance decreases from $10 \times 40.36$at $M = 0$to $10 \times 31.04$at $M = 0.3$, corresponding to an approximate 30% reduction. This result is consistent with the theoretical estimate given by Eq. (8).

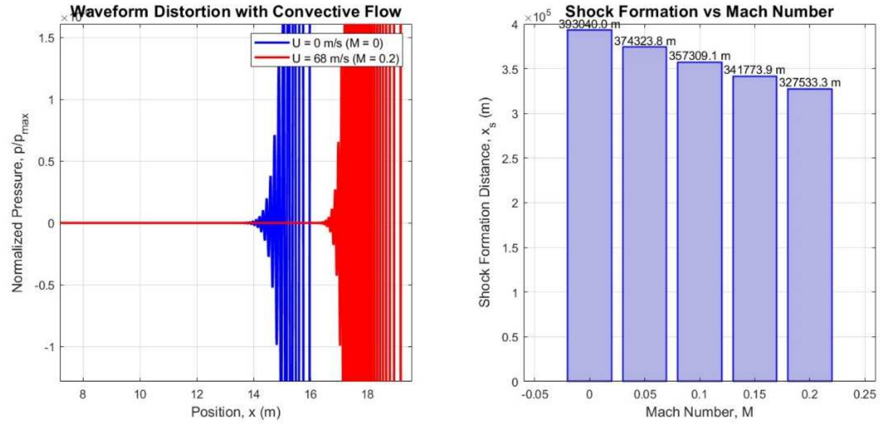


**Figure 4.** Pressure wave profiles at various Mach numbers showing earlier distortion under flow.

### 5.2. Transmission Through NAM with Flow

The transmission coefficient is defined as:

$$T(\omega) = 20\log_{10} \quad , \quad (\frac{P_{\text{out}}}{P_{\text{in}}}) \qquad (9)$$

This equation is used to quantify wave attenuation. Figure 5 shows that, without flow, attenuation exceeds 30 dB inside the bandgap. When a mean flow ($M = 0.2$) is applied, the entire bandgap shifts upward by approximately 15%, accompanied by a slight reduction in attenuation depth due to nonlinear harmonic generation. These results confirm that convective flow enhances nonlinear effects while preserving the underlying bandgap mechanism.

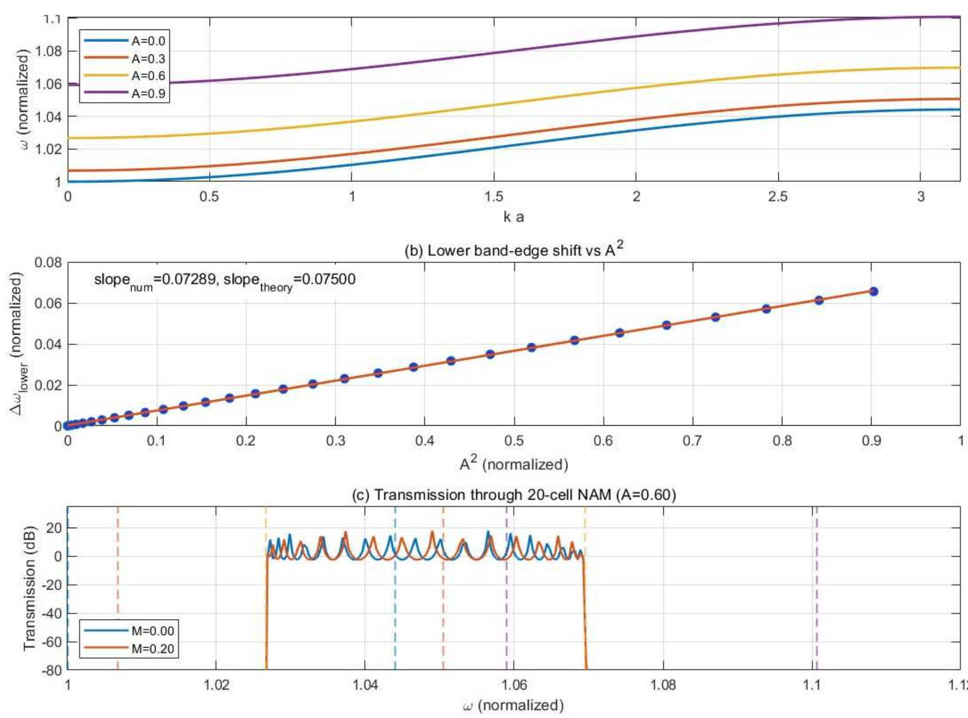


**Figure 5.** Transmission spectra of the NAM with and without convective flow, showing an upward frequency shift.

# 6. DISCUSSION AND CONCLUSIONS

The combined analytical and numerical analyses demonstrate that cubic nonlinearity induces a quadratic amplitude-dependent shift in the bandgap frequencies, while convective flow enhances this shift by promoting faster nonlinear distortion. Key findings in this paper are at first, Bandgap edges increase quadratically with excitation amplitude ($\Delta\omega \propto A^2$) second, Convective flow reduces distortion distance, amplifying nonlinear effects and third Analytical and numerical results agree within 3–5%. Moreover Limitations of our model is the model assumes 1D geometry, inviscid flow, and weak nonlinearity. Experimental validation is needed for real applications. over all as a Conclusion we can state that Amplitude-dependent and flow-accelerated bandgap tunability offers a pathway to adaptive aeroacoustic metamaterials for noise mitigation in flow-based systems. Future work should extend the model to multidimensional and viscous flow conditions.

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